On a J-polar decomposition of a bounded operator and matrix representations of J-symmetric, J-skew-symmetric operators.

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Introduction.

Complex symmetric, skew-symmetric and orthogonal matrices are classical objects of the finite-dimensional linear analysis [1]. In particular, the canonical spectral forms are known for them. Certainly, they have a more complicated structures as for Hermitian matrices. However, in a certain sense complex symmetric matrices are more universal. Namely, an *arbitrary* square complex matrix is similar to a symmetric matrix. If one introduces a J-form and write conditions for a symmetric, skew-symmetric and orthogonal matrix (continued by zeros to the right and to the bottom to obtain a semi-infinite matrix) in its terms, one arrives to the well-known J-symmetric, J-skew-symmetric and J-isometric operators.

A general definition of a J-symmetric operator was given by I.M. Glazman in his paper [2]. A study of these operators had been continued in papers of N.A. Zhyhar and A. Galindo (see the references in a monograph [3]). Later, an investigation of these operators had been performed by A.D. Makarova, L.A. Kamerina, V.P. Li, T.B. Kalinina, A.N. Kochubey, B.G. Mironov (a seria of papers by these authors appeared in 70-th, 80-th of the 20-th century in Ulyanovskiy sbornik "Funkcionalniy analiz"), L.M. Rayh, E.R. Tsekanovskiy and others. Most of these papers were devoted to the questions of extensions of J-symmetric operators to J-self-adjoint operators and to a description of all such extensions. At the present time, J-self-adjoint operators are studied by S.R. Garcia, M. Putinar, E. Prodan (see the paper [4] and References therein).

A definition of a bounded J-skew-symmetric operator was given by Sh. Asadi and I.E. Lutsenko in the paper [5]. A general definition appeared in a paper of T.B. Kalinina [6], she continued to study these operators in papers [7], [8]. J-symmetric and J-skew-symmetric operators also appeared in a book [9] in a study of Volterra operators context.

In papers of L.A. Kamerina J-isometric and quasi-unitary operators and a notion of quasi-unitary equivalence were introduced [10],[11].

Consider a separable Hilbert space H. Recall that a conjugation (involution) in H is an operator J, defined on the whole H and satisfying the following properties [12],[13]

$$J^2 = E, \qquad (Jx, Jy) = \overline{(x,y)}, \qquad x, y \in H, \tag{1}$$

where E is the identity operator in H, and $(\cdot,\cdot)$ is a scalar product in H. For each conjugation there exists an orthonormal basis $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ in H such that

$$Jx = \sum_{k=0}^{\infty} \overline{x_k} f_k, \qquad x = \sum_{k=0}^{\infty} x_k f_k \in H. \tag{2}$$

This basis is not uniquely determined, it is determined up to a unitary transformation which commutes with J (J-real). An arbitrary such a basis $\mathcal{F}$ we shall call **corresponding** to the involution J. Define the following linear with respect to the both arguments functional (J-form):

$$[x,y]_J := (x, Jy), \qquad x,y \in H. \tag{3}$$

A linear operator A in H is said to be J-symmetric, if

$$[Ax,y]_J = [x,Ay]_J, \qquad x,y \in D(A), \tag{4}$$

and is said to be J-skew-symmetric if

$$[Ax,y]_J = -[x,Ay]_J, \qquad x,y \in D(A). \tag{5}$$

If the following condition is true:

$$[Ax,Ay]_J = [x,y]_J, \qquad x,y \in D(A), \tag{6}$$

then the operator is said to be J-isometric.

Let the domain of A is dense in H. The operator A is said to be J-self-adjoint if

$$A = JA^*J, \tag{7}$$

and is said to be J-skew-self-adjoint if

$$A = -JA^*J. \tag{8}$$

If

$$A^{-1} = JA^*J, \tag{9}$$

then the operator A we shall call a J-unitary. Notice that the operator $A^T = JA^*J$ in [12] was called **transposed** (later, in some papers it was also called J-adjoint, but we shall use the latter word for the operator $\widetilde{A} = JAJ$).

For non-densely defined operators, one can also introduce a notion of J-symmetric and J-skew-symmetric linear relations, see, e.g., [14].

Let A be a linear bounded operator in H. In this case, conditions (4),(5), (6) mean that the matrix of the operator in an arbitrary basis $\mathcal{F}$, which is

coorresponding to J, will be symmetric, skew-symmetric and orthogonal, respectively. This remark and some properties of the J-form allow to obtain some simple properties of eigenvalues and eigenvectors of such matrices.

In this work we obtain a J-polar decomposition for bounded operators (under some conditions). This decomposition is analogous to the polar decomposition of a bounded operator and to the J-polar decomposition in J-spaces [15]. Also, we obtain other decompositions which are analogous to decompositions for finite-dimensional matrices in [1]. A possibility of the matrix representation for J-symmetric and J-skew-symmetric operators and its properties are studied. A structure of the following null set $H_{J;0} = \{x \in H : [x,x]_J = 0\}$, is studied, as well.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}^2$ the sets of real numbers, complex numbers, positive integers, non-negative integers and the real plane, respectively. Everywhere in this paper, all Hilbert spaces are assumed to be separable, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in a Hilbert space, respectively.

For a set M in a Hilbert space H, by $\overline{M}$ we mean a closure of M in the norm $\|\cdot\|$. For $\{x_k\}_{k\in\mathbb{Z}_+}$, $x_k \in H$, we write $\mathrm{Lin}\{x_k\}_{k\in\mathbb{Z}_+} := \{y \in H : \ y = \sum_{j=0}^n \alpha_j x_j, \ \alpha_j \in \mathbb{C}, \ n \in \mathbb{Z}_+\}$; $\mathrm{span}\{x_k\}_{k\in\mathbb{Z}_+} := \overline{\mathrm{Lin}\{x_k\}_{k\in\mathbb{Z}_+}}$.

The identity operator in a Hilbert space H is denoted by E. For an arbitrary linear operator A in H, the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A, and by $\mathrm{Ker}\, A$ we mean the kernel of the operator A. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of A and the resolvent set of A, respectively. The resolvent function of A we denote by $R_\lambda(A)$, $\lambda \in \rho(A)$. Also, we denote $\Delta_A(\lambda) = (A - \lambda E)D(A)$. The norm of a bounded operator A is denoted by $\|A\|$.

By l_2 we denote the space of complex sequences $x = (x_0, x_1, x_2, ...)^T$, $x_k \in \mathbb{C}$, $k \in \mathbb{Z}_+$, with a finite norm $\|x\| = \left(\sum_{k=0}^\infty |x_k|^2\right)^{\frac{1}{2}}$ (the superscript T stands for the transposition).

1 Properties of eigenvalues and eigenvectors.

We shall begin with some simple properties of J-symmetric, J-skew-symmetric and J-orthogonal operators which, in particular, lead to some new properties of finite-dimensional complex symmetric, skew-symmetric and orthogonal matrices. Let J be a conjugation in a Hilbert space H.

Vectors x and y are said to be **J-orthogonal**, if $[x,y]_J = 0$. The following proposition is true (concerning statement (i) of the Proposition see. The-

orem 2 in a paper [16, p.86]).

Proposition 1.1 *Let A be a J-symmetric operator in a Hilbert space H. The following statements are true:*
(i) *Eigenvectors of the operator A which correspond to different eigenvalues are J-orthogonal;*
(ii) *If vectors x and Jx, $x \in D(A)$, are eigenvectors of the operator A, then they correspond to the same eigenvalue.*

Proof. In fact, we can write

$$\lambda_x[x, y]_J = [Ax, y]_J = [x, Ay]_J = \lambda_y[x, y]_J,$$

and therefore

$$(\lambda_x - \lambda_y)[x, y]_J = 0. \tag{10}$$

Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator A, which correspond to eigenvalues λ_x and $\lambda_{\overline{x}}$, respectively. Write (10) with $y = \overline{x}, \lambda_y = \lambda_{\overline{x}}$:

$$(\lambda_x - \lambda_{\overline{x}})[x, \overline{x}]_J = 0.$$

Since $[x, \overline{x}]_J = \|x\|^2 > 0$, we get $\lambda_x = \lambda_{\overline{x}}$. $\square$

Define the following set:

$$H_{J;0} := \{x \in H : \ [x, x]_J = 0\}. \tag{11}$$

In a similar to the latter proof manner the validity of the following two propositions is established.

Proposition 1.2 *If A is a J-skew-symmetric operator in a Hilbert space H, then the following is true:*
(i) *Eigenvectors of the operator A, which correspond to non-zero eigenvalues, belong to the set $H_{J;0}$;*
(ii) *If λ_x, λ_y are eigenvalues of the operator A such that $\lambda_x \neq -\lambda_y$, then the corresponding to them eigenvectors are J-orthogonal;*
(iii) *Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator A, corresponding to the eigenvalues λ_x and $\lambda_{\overline{x}}$, respectively. Then $\lambda_x = -\lambda_{\overline{x}}$.*

Proposition 1.3 *Let A be a J-isometric operator in a Hilbert space H. Then the following statements are true:*
(i) *Eigenvectors of the operator A, which correspond to different from ± 1 eigenvalues belong to the set $H_{J;0}$;*

(ii) *If* λ_x, λ_y *are eigenvalues of the operator* A *such that* $\lambda_x \neq \frac{1}{\lambda_y}$, *then the corresponding to them eigenvectors are J-orthogonal;*
(iii) *Suppose that* $x, \overline{x} := Jx \in D(A)$ *are eigenvectors of the operator* A, *corresponding to the eigenvalues* λ_x *and* $\lambda_{\overline{x}}$, *respectively. Then* $\lambda_x = \frac{1}{\lambda_{\overline{x}}}$.

It is interesting to notice that in the finite-dimensional case the point 0 for a skew-symmetric matrix and points ± 1 for an orthogonal matrix are distinguished in a special manner in the spectrum, as well.

In the case of a unitary space U^n with a dimension n, $n \in \mathbb{Z}_+$, in an analogous manner, a conjugation J, a J-form, and J-orthogonality are defined. So, the latter statements are true for complex symmetric, skew-symmetric and orthogonal matrices.

Example 1.1. Consider a numerical matrix $A = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$, and the corresponding to them normed eigenvectors are $f_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} - i \\ 2 \end{pmatrix}$, $f_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{3} - i \\ 2 \end{pmatrix}$. Vectors f_1, f_2 are not orthogonal. However, they are J-orthogonal.

Let J be a conjugation in a Hilbert space H and A be a bounded linear operator in H. The norm of A, as it can be easily seen from the properties of the involution, can be calculated by the following formula

$$\|A\| = \sup_{x,y \in H:\ \|x\|=\|y\|=1} |[Ax, y]_J|. \tag{12}$$

The following statement is true:

Proposition 1.4 *If* A *is a bounded J-symmetric operator in a Hilbert space* H, *then its norm can be calculated as*

$$\|A\| = \sup_{x \in H:\ \|x\|=1} |[Ax, x]_J|. \tag{13}$$

Proof. Consider an operator A such as in the statement of the Proposition. Set $C := \sup_{x \in H:\ \|x\|=1} |[Ax, x]_J|$. For arbitrary elements $x, y \in H :\ x \neq \pm y$ we can write

$$[A(x+y), x+y]_J - [A(x-y), x-y]_J = 4[Ax, y]_J;$$

$$|[Ax, y]_J| \leq \frac{1}{4} \left(|[A(x+y), x+y]_J| + |[A(x-y), x-y]_J| \right) =$$

$$= \frac{1}{4} \Big(\left| \left[A\Big(\frac{x+y}{\|x+y\|}\Big), \frac{x+y}{\|x+y\|} \right]_J \right| \|x+y\|^2 + \left| \left[A\Big(\frac{x-y}{\|x-y\|}\Big), \frac{x-y}{\|x-y\|} \right]_J \right| *$$

$$* \|x-y\|^2) \leq \frac{1}{4}C(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2}C(\|x\|^2 + \|y\|^2). \tag{14}$$

Thus, by using (12) and (14) we get

$$\|A\| = \sup_{x,y\in H:\ \|x\|=\|y\|=1} |[Ax, y]_J| \leq C.$$

On the other hand, we can write

$$C = \sup_{x,y\in H:\ \|x\|=1} |[Ax, x]_J| \leq \sup_{x,y\in H:\ \|x\|=\|y\|=1} |[Ax, y]_J| = \|A\|.$$

Therefore $C = \|A\|$. $\Box$

For a J-skew-symmetric operator A, its norm can not be calculated by the formula (13). Moreover, the following characteristic property of J-skew-symmetric operators is true.

Proposition 1.5 *A linear operator A in a Hilbert space H is J-skew-symmetric if and only if the following equality is true*

$$[Ax, x]_J = 0, \qquad x \in D(A). \tag{15}$$

Proof. We first notice that from the properties of an involution it follows that $[x, y]_J = [y, x]_J, \quad x, y \in H$. Let us check the necessity. From relation (5) it follows that

$$[Ax, x]_J = -[x, Ax]_J = -[Ax, x]_J,$$

and therefore (15) holds true.
Let us check the sufficiency. By using (15) we write

$$0 = [A(x+y), x+y]_J = [Ax, x]_J + [Ax, y]_J + [Ay, x]_J + [Ay, y]_J =$$

$$= [Ax, y]_J + [Ay, x]_J, \qquad x, y \in D(A).$$

From this relation we obtain that $[Ax, y]_J = -[Ay, x]_J = -[x, Ay]_J$. $\Box$

Let J be a conjugation in a Hilbert space H and A be an arbitrary linear operator in H. The operator $\widetilde{A} := \widetilde{(A)}_J := JAJ$ we shall call **J-adjoint** to the operator A. We first note that $\widetilde{\widetilde{A}} = A$ and the following easy to check lemma is true.

Lemma 1.1 *For a linear operator A in a Hilbert space H, equalities $\overline{D(A)} = H$ and $\overline{D(\widetilde{A})} = H$ are true or false simultaneously. The same can be said about equalities $\overline{R(A)} = H$ and $\overline{R(\widetilde{A})} = H$.*

An operation of the construction of the J-adjoint operator commutes with basic operations on operators. Let us formulate the necessary for us properties as propositions.

Proposition 1.6 *Let A be a linear operator in a Hilbert space H such that $\overline{D(A)} = H$ and J be a conjugation in H. Then the following relation is true*

$$\widetilde{A^*} = (\widetilde{A})^*. \tag{16}$$

Proof. Choose an arbitrary element $g \in D((\widetilde{A})^*)$. On one hand, it is true

$$(\widetilde{A}x, g) = (x, (\widetilde{A})^* g) = (JJx, JJ(\widetilde{A})^* g) = \overline{(Jx, J(\widetilde{A})^* g)} =$$

$$= (J(\widetilde{A})^* g, Jx), \qquad x \in D(\widetilde{A}).$$

On the other hand, wee can write

$$(\widetilde{A}x, g) = (JAJx, JJg) = \overline{(AJx, Jg)} = (Jg, AJx), \qquad x \in D(\widetilde{A}).$$

Comparing right hand sides we obtain that

$$(AJx, Jg) = (Jx, J(\widetilde{A})^* g),$$

and therefore $Jg \in D(A^*)$, $A^* Jg = J(\widetilde{A})^* g$. Multiplying by J both sides of the latterr equality we get $\widetilde{A^*} g = (\widetilde{A})^* g$. Therefore

$$(\widetilde{A})^* \subseteq \widetilde{A^*}. \tag{17}$$

In order to obtain the inverse inclusion one should write the inclusion (17) with the operator $\widetilde{A}$, and then to take J-adjoint operators for the both sides (the inclusion under the last operation will stay true). □

Proposition 1.7 *Let A be a linear operator in a Hilbert space H and J be a conjugation in H. Suppose that operators A and $\widetilde{A}$ admit closures. Then the following equality is true*

$$\widetilde{\overline{A}} = \overline{\widetilde{A}}. \tag{18}$$

Proof. Choose an arbitrary element $g \in D(\overline{\widetilde{A}})$. Then there exists a sequence $x_n \in D(\widetilde{A}), n \in \mathbb{Z}_+$, such that $x_n \to x$, $\widetilde{A}x_n = JAJx_n \to \overline{\widetilde{A}}x$ when $n \to \infty$. By continuity of the operator J from this it follows that

$$Jx_n \to Jx, \quad AJx_n \to J\overline{\widetilde{A}}x.$$

Consequently, we obtain, that $Jx \in D(\overline{A})$ $\overline{A}Jx = J\overline{\widetilde{A}}x$. Therefore $x \in D(\widetilde{\overline{A}})$ and $\widetilde{\overline{A}}x = \overline{\widetilde{A}}x$. From this relation we conclude that

$$\overline{\widetilde{A}} \subseteq \widetilde{\overline{A}}. \tag{19}$$

In order to obtain the inverse inclusion, we write the inclusion (19) for the operator $\widetilde{A}$, and then to take J-adjoint operators for the both sides. □

Proposition 1.8 *Let A be a linear invertible operator in a Hilbert space H and J be a conjugation in H. Then the operator $\widetilde{A}$ is also invertible and the following equality is true*

$$\widetilde{A^{-1}} = (\widetilde{A})^{-1}. \tag{20}$$

Proof. Since $\widetilde{A^{-1}}\widetilde{A} = E|_{D(\widetilde{A})}$, and $D(\widetilde{A^{-1}}) = JD(A^{-1}) = JR(A) = R(\widetilde{A})$, the operator $\widetilde{A}$ is invertible and relation (20) is true. □

Notice that a condition of a J-symmetric operator (4) with the help of J-adjoint operator will be written as follows:

$$(Ax, y) = (x, \widetilde{A}y), \qquad x \in D(A),\ y \in D(\widetilde{A}). \tag{21}$$

Conditions of a J-skew-symmetric operator (5) and J-isometric operator(6) will be written as

$$(Ax, y) = -(x, \widetilde{A}y), \qquad x \in D(A),\ y \in D(\widetilde{A}), \tag{22}$$

and

$$(Ax, \widetilde{A}y) = (x, y), \qquad x \in D(A),\ y \in D(\widetilde{A}), \tag{23}$$

respectively.

Now we shall assume that the operator A is densely defined in H. Notice that in this case from condition (23) it follows that the operator A is invertible. In fact, equality $Ax = 0$ implies the equality $(x, y) = 0$ on a dense in H set $D(\widetilde{A})$. Thus, a densely defined J-isometric operator is always invertible.

Note that in the case of a densely defined operator A, conditions (21),(22),(23) are equivalent to the following conditions

$$A \subseteq (\widetilde{A})^{*}, \tag{24}$$

$$A \subseteq -(\widetilde{A})^{*}, \tag{25}$$

and

$$A^{-1} \subseteq (\widetilde{A})^{*}, \tag{26}$$

respectively. From these relations, in particular, it immediately follows that densely defined J-symmetric and J-skew-symmetric operators admit closures. As it is seen from relations (4),(5), their closures will also be J-symmetric or J-skew-symmetric operators, respectively. For a densely defined J-isometric operator one can only state that its inverse operator admits a closure. However, from relation (6) it is easily seen that the inverse operator to a J-isometric is also J-isometric. Consequently, if the range of the original J-isometric operator (the domain of the inverse operator) is also dense, then it admits a closure. In this case, also from the relation (6), it is seen that this closure will be a J-isometric operator.

Note that the operation of the construction of a J-adjoint operator does not change the defined above by us types of operators. Namely, the following proposition is true:

Proposition 1.9 *Let A be a linear operator in a Hilbert space H and J be a conjugation in H. If the operator A is J-symmetric, J-skew-symmetric or J-isometric, then the same is the operator $\widetilde{A} = JAJ$, as well.*

Proof. The statement about a J-symmetric (J-skew-symmetric, J-isometric) operator follows from relation (21) ((22), (23)), respectively, taking into account that $A = \widetilde{\widetilde{A}}$. $\Box$

For an element $x \in H$ and a set $M \subseteq H$ we write $x \perp_J M$, if $x \perp_J y$, forr all $y \in M$. For a set $M \subseteq H$ we denote $M_J^{\perp} = \{x \in H : \ x \perp_J y, \ y \in M\}$.

It is known that the residual spectrum of a J-self-adjoint operator is empty. It follows from the theorem below.

Theorem 1.1 *([16, Theorem 4, p.87]) Let A -be a J-self-adjoint operator in a Hilbert space H. A complex number λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A(\lambda)} \neq H. \tag{27}$$

In this case, $(\Delta_A(\lambda))_J^{\perp}$ will be an eigen-subspace which corresponds to λ.

We shall obtain analogous results for J-skew-symmetric and J-isometric operators. The following theorem is true:

Theorem 1.2 *Let A be a J-skew-self-adjoint operator in a Hilbert space H. A complex value λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A(-\lambda)} \neq H. \tag{28}$$

In this case, $(\Delta_A(-\lambda))_J^{\perp}$ will be an eigen-subspace which corresponds to λ.

Proof. *Necessity.* Let x be an arbitrary eigenvector of the operator A which corresponds to an eigenvalue λ. Since A, in particular, is skew-symmetric, then we can write for an arbitrary $y \in D(A)$

$$0 = [(A - \lambda E)x, y]_J = -[x, (A + \lambda E)y]_J. \tag{29}$$

Therefore $x \perp_J \Delta_A(-\lambda)$ and by the continuity of $[\cdot,\cdot]_J$ we get

$$x \perp_J \overline{\Delta_A(-\lambda)}. \tag{30}$$

Since $[x, Jx] = \|x\|^2 > 0$, then $Jx \notin \overline{\Delta_A(-\lambda)}$ and therefore $\overline{\Delta_A(\lambda)} \neq H$.
Sufficiency. Suppose that equality (28) is true. Then there exists $0 \neq y \in H$ such that

$$(z, y) = 0, \qquad z \in \overline{\Delta_A(-\lambda)}. \tag{31}$$

Therefore $((A + \lambda E)x, y) = 0$, and from this relation we get $(Ax, y) = (x, \overline{(-\lambda)}y)$, $x \in D(A)$. Thus, we have $y \in D(A^*)$ and

$$A^* y = -\overline{\lambda} y. \tag{32}$$

But since A is J-skew-self-adjoint, then $A^* = -\widetilde{A}$, and we obtain

$$\widetilde{A} y = \overline{\lambda} y.$$

From this relation it follows that $Jy \neq 0$ is an eigenvector of the operator A with an eigenvalue λ.

Let us show that the following set

$$V(\lambda) := (\Delta_A(-\lambda))^{\perp}_J \setminus \{0\}, \tag{33}$$

is a set of eigenvectors of the operator A, corresponding to a eigenvalue λ. Denote the latter set by $S(\lambda)$. By the proven property (30), the inclusion $S(\lambda) \subseteq V(\lambda)$ is true. On the other hand, if $x \in V(\lambda)$, then for $y := Jx$ relation (31) is true. Repeating arguments which follow after this formula we come to a conclusion that x is an eigenvector of the operator A, corresponding to λ. Thus, the inverse inclusion is also true.

Finally, since $A = (\widetilde{A})^*$, then A is closed. Therefore $(\Delta_A(-\lambda))^{\perp}_J$ is an eigen-subspace of the operator A, which corresponds to λ. $\square$

Corollary 1.1 *The point* 0 *can not belong to the residual spectrum of a J-skew-self-adjoint operator.*

In an analogous manner, the following result for J-unitary operators is established.

Theorem 1.3 *Let A be a J-unitary operator in a Hilbert space H. A complex number λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A\left(\frac{1}{\lambda}\right)} \neq H. \tag{34}$$

In this case, $(\Delta_A(\frac{1}{\lambda}))^{\perp}_J$ is an eigen-subspace, which corresponds to λ.

Corollary 1.2 *Points ± 1 can not belong to the residual spectrum of a J-unitary operator.*

From relations (21),(22) it is seen that a defined in the whole H J-symmetric (J-skew-symmetric) operator is a bounded J-self-adjoint (respectively J-skew-swlf-adjoint) operator. The following statements are also true.

Proposition 1.10 *([16, Theorem 1, p.85-86],[6, Theorem 3, p.69]) Let A be a linear densely defined operator in a Hilbert space H, which is J-symmetric (J-skew-symmetric). Suppose that $R(A) = H$. Then the operator A is a J-self-adjoint (respectively J-skew-self-adjoint) operator.*

Proposition 1.11 *Let A be a linear densely defined operator in a Hilbert space H, which is J-symmetric (J-skew-symmetric). Suppose that $\overline{R(A)} = H$. Then the operator A is invertible and the operator A^{-1} is also a J-symmetric (respectively J-skew-symmetric) operator.*

Proof. In a view of analogous considerations, we shall check the validity of this Proposition only for the case of a J-skew-symmetric operator A. Notice that $\mathrm{Ker}\, A^* = H \ominus \overline{R(A)} = \{0\}$. Thus, the operator A^* is invertible. Since A is J-skew-symmetric, the following inclusion is true $\widetilde{A} \subseteq -A^*$ and therefore $\widetilde{A}$ is invertible, as well. By Proposition 1.8 we conclude that the operator A has an inverse operator. From the inclusion $\widetilde{A} \subseteq -A^*$ it follows the following inclusion

$$(\widetilde{A})^{-1} \subseteq -(A^*)^{-1}. \tag{35}$$

Notice that $\overline{D(A^{-1})} = \overline{R(A)} = H$. Thus, we can state that $(A^*)^{-1} = (A^{-1})^*$. Using this equality and using Proposition 1.8, from relation (35) we obtain the following inclusion

$$\widetilde{A^{-1}} \subseteq -(A^{-1})^*.$$

And this means that the operator A^{-1} is J-skew-symmetric. $\square$

Proposition 1.12 *Let A be a J-self-adjoint (J-skew-self-adjoint) operator in a Hilbert space H. Suppose that $\overline{R(A)} = H$. Then the operator A is invertible and the operator A^{-1} is also J-self-adjoint (respectively J-skew-self-adjoint) operator.*

Proof. In a view of analogous considerations, we shall give the proof only for the case of J-self-adjoint operator A. By Proposition 1.11 the operator A is invertible. By Proposition 1.8 the operator $\widetilde{A}$ is invertible, as well. From Lemma 1.1 it follows that $\overline{R(\widetilde{A})} = H$ $\overline{D(\widetilde{A})} = H$. Thus, we have $\overline{D((\widetilde{A})^{-1})} = H$. Consequeently, the following equality is true $((\widetilde{A})^{*})^{-1} = ((\widetilde{A})^{-1})^{*}$. Since the operator A is J-self-adjoint, the last equality can be written as $A^{-1} = ((\widetilde{A})^{-1})^{*}$. Using Proposition 1.8, we obtain the following equality $A^{-1} = (\widetilde{A^{-1}})^{*}$, which shows that the operator A^{-1} is J-self-adjoint. □

2 A J-polar decomposition of bounded operators.

We shall extend in the case of J-symmetric, J-skew-symmetric and J-isometric operators a seria of properties of finite-dimensional complex symmetric, skew-symmetric and orthogonal matrices (see [1]).

The following lemma is true:

Lemma 2.1 *Let A be a bounded self-adjoint and J-isometric operator in a Hilbert space H. Then the operator A admits the following representation:*

$$A = Ie^{iK}, \tag{36}$$

where I is a bounded self-adjoint J-real involutory ($I^2 = E$) operator in H, and K is a commuting with I bounded skew-self-adjoint J-real operator in H.

If additionally it is known that the operator A is positive, $A \geq 0$, then one can choose $I = E$.

Proof. Consider an operator A such as in the statement of the Lemma. Since the operator A is J-isometric and bounded, then from (6) we obtain $A^{*}JA = J$, or $A^{*}\widetilde{A} = E$. Since A is self-adjoint, then

$$A\widetilde{A} = E. \tag{37}$$

For the operator A we can write the following representation

$$A = S + iT, \tag{38}$$

where $S = \frac{1}{2}(A + \widetilde{A})$, $T = \frac{1}{2i}(A - \widetilde{A})$. By this, operators S and T are J-real, the operator S is self-adjoint and J-self-adjoint, and the operator T is skew-self-adjoint and J-skew-self-adjoint. Since $\widetilde{A} = S - iT$, then from relation (37) we get

$$E = A\widetilde{A} = (S + iT)(S - iT) = S^2 + T^2 + i(TS - ST).$$

From this relation it follows that operators T and S commute and

$$S^2 + T^2 = E. \tag{39}$$

Since operators S and iT are commuting bounded self-adjoint operators, then they admit spectral representations

$$S = \int_L \lambda dE_\lambda, \qquad iT = \int_L z dF_z, \tag{40}$$

where E_λ, F_z are commuting resolutions of unity of the operators, and $L = (l_1, l_2]$, $l_1, l_2 \in \mathbb{R}$, is a finite interval of the real line which contains the spectra of operators. From equality (39), by using spectral resolutions we get

$$\int_L \int_L (\lambda^2 - z^2 - 1) dE_\lambda dF_z = 0, \tag{41}$$

where the integral means a limit in the norm of H of the corresponding Riemann-Stieltjes type sums (in the plane).

A point $(\lambda_0, z_0) \in \mathbb{R}^2$ we call **a point of increase** for the measure $dE_\lambda dF_z$, if for an arbitrary number $\varepsilon > 0$, there exists an element $x \in H$ such that

$$(E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(F_{z_0+\varepsilon} - F_{z_0-\varepsilon})x \neq 0, \tag{42}$$

or, equivalently,

$$((E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(F_{z_0+\varepsilon} - F_{z_0-\varepsilon})x, x) > 0. \tag{43}$$

For an arbitrary point of increase $(\lambda_0, z_0) \in \mathbb{R}^2$ of the measure $dE_\lambda dF_z$ it is true

$$\lambda_0^2 - z_0^2 - 1 = 0. \tag{44}$$

In fact, if the latter equality is not true for a point of increase $u_0 = (\lambda_0, z_0) \in \mathbb{R}^2$, then $|\lambda^2 - z^2 - 1| \geq a$, $a > 0$, in a neighborhood $U = U(\lambda_0, z_0; \varepsilon) = \{(\lambda, z) \in \mathbb{R}^2 : \lambda_0 - \varepsilon < \lambda \leq \lambda_0 + \varepsilon, z_0 - \varepsilon < z \leq z_0 + \varepsilon\}$, $\varepsilon > 0$, of the point

u_0. For this number ε, there exists an element $x \in H$ such that (43) is true. But

$$0 = \| \int_L \int_L (\lambda^2 - z^2 - 1) dE_\lambda dF_z x \|^2 = \int_L \int_L |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x) \geq$$

$$\geq \int \int_U |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x) \geq a^2((E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(E_{z_0+\varepsilon} - E_{z_0-\varepsilon})x, x) > 0.$$

If two continuous functions $\varphi(\lambda, z)$ $\widehat{\varphi}(\lambda, z)$ on $L^2 = \{(\lambda, z) \in \mathbb{R}^2 : \ \lambda, z \in L\}$ coincide in the points of increase of the measure $dE_\lambda dF_z$, then

$$\int_L \int_L \varphi(\lambda, z) dE_\lambda dF_z = \int_L \int_L \widehat{\varphi}(\lambda, z) dE_\lambda dF_z. \tag{45}$$

In fact,

$$\| \int_L \int_L (\varphi(\lambda, z) - \widehat{\varphi}(\lambda, z)) dE_\lambda dF_z x \|^2 = \int_L \int_L |\varphi(\lambda, z) - \widehat{\varphi}(\lambda, z)|^2 (dE_\lambda dF_z x, x),$$

and it remains to notice that $(dE_\lambda dF_z x, x)$ is a positive measure on L^2, and the function under the integral is equal to zero in all points of increase of this measure.

Consider a set $\Gamma \subset \mathbb{R}^2$, which consists of points $(\lambda, z) \in \mathbb{R}^2$, such that

$$\lambda^2 - z^2 - 1 = 0. \tag{46}$$

From (46) it follows that for all poiints of the set Γ it is true $|\lambda| = \sqrt{1 + z^2}$ (where we mean the arithmetic value of the root). Hence, for all points of Γ

$$\lambda = \operatorname{sgn}(\lambda) \sqrt{1 + z^2}, \tag{47}$$

where

$$\operatorname{sgn}(\lambda) = \left\{ \begin{array}{ll} 1, & \lambda > 0, \\ -1, & \lambda \leq 0 \end{array} \right. . \tag{48}$$

By the identity $z = \operatorname{sh} \operatorname{arcsh} z$, the equality (47) can be rewritten in the following form

$$\lambda = \operatorname{sgn}(\lambda) \sqrt{\operatorname{ch}^2(\operatorname{arcsh} z)} = \operatorname{sgn}(\lambda) \operatorname{ch}(\operatorname{arcsh} z), \tag{49}$$

in a view of positivity of the hyperbolic cosine function. By this representation we can write

$$A = S + iT = \int_L \int_L (\lambda + z) dE_\lambda dF_z = \int_L \int_L (\operatorname{sgn}(\lambda) \operatorname{ch}(\operatorname{arcsh} z) + z) dE_\lambda dF_z =$$

$$= \int_L \int_{L_+} e^{\operatorname{arcsh} z} dE_\lambda dF_z + \int_L \int_{L_-} (-e^{-\operatorname{arcsh} z}) dE_\lambda dF_z, \tag{50}$$

where $L_+ = (0, \infty) \cap L,\ L_- = (-\infty, 0] \cap L$.

Define the following operator

$$V = \int_L \int_L \operatorname{sgn}(\lambda) \operatorname{arcsh} z dE_\lambda dF_z = \int_L \operatorname{sgn}(\lambda) dE_\lambda \int_L \operatorname{arcsh} z dF_z. \tag{51}$$

The operator V is bounded self-adjoint and J-imaginary. In fact, since the operator S is J-real, then its resolution of unity E_λ commutes with J (see [13]). Therefore the operator

$$I := \int_L \operatorname{sgn}(\lambda) dE_\lambda, \tag{52}$$

is a bounded J-real self-adjoint involutory operator. On the other hand, $\operatorname{arcsh}(iT) = \sum_{k=0}^{\infty} a_{2k+1} (iT)^{2k+1}$, $a_{2k+1} \in \mathbb{R}$, is a J-imaginary, as a limit of J-imaginary operators (here the convergence is understood in the norm of H).

From relations (50),(51),(52) we conclude that

$$A = Ie^V.$$

Set $K = -iV$, and we obtain the required representation (36).

If it is additionally known that the operator A is positive, $A \geq 0$, then

$$I = Ae^{-V} = (e^{-\frac{V}{2}})^* A e^{-\frac{V}{2}},$$

is positive, as well. Therefore I is a positive square root of E. By the uniqueness of such a root we conclude that $I = E$. $\square$

The following theorem is true.

Theorem 2.1 *Let A be a bounded J-unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A = Re^{iK}, \tag{53}$$

where R is J-real unitary operator in H, and K is a bounded J-real skew-self-adjoint operator in H.

Proof. Consider an operator A such as in the statement of the Theorem. Suppose that representation (53) is true. Then

$$A^* A = e^{iK} R^* R e^{iK} = e^{2iK}.$$

Now we shall do not assume an existence of representation (53) and notice that the operator $G := A^*A$ is positive self-adjoint and J-unitary. In fact, since the operator A is bounded by assumption and J-unitary, then A^* is also bounded and J-unitary. A product of bounded J-unitary operators is a bounded J-unitary operator, this is verified directly by the definition. By Lemma 2.1 we find a bounded J-real skew-self-adjoint operator K such that

$$G = e^{2iK}. \tag{54}$$

Now set by definition

$$R = Ae^{-iK}. \tag{55}$$

By equality (54) we can write

$$R^*R = e^{-iK}A^*Ae^{-iK} = E,$$

and, hence, the operator R is unitary. Now notice that

$$Je^{-iK}J = J(\cos(iK) - i\sin(iK))J = \cos(iK) + i\sin(iK),$$

since the operator iK is J-real and therefore its resolution of unity commutes with J. Consequently, we have

$$Je^{-iK}J = e^{iK} = (e^{-iK})^{-1}, \tag{56}$$

and the operator e^{-iK} is J-unitary. By (55),(56) and using that the operator A is J-unitary we conclude that

$$R^{-1} = e^{iK}A^{-1} = Je^{-iK}JJA^*J = J(Ae^{-iK})^*J = \widetilde{(R^*)},$$

and therefore the operator R is J-unitary. Then $R^{-1} = R^* = JR^*J$, and therefore R^* is a J-real operator. Using matrix representations of operators R^* and R in an arbitrary basis, which corresponds to the involution J, we conclude that the operator R is J-real. $\Box$

Lemma 2.2 *Let A be a J-self-adjoint and unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A = e^{iS}, \tag{57}$$

where S is a bounded J-real self-adjoint operator in H.

Proof. Consider an operator A such as in the statement of the Lemma. For the J-self-adjoint operator A it is true $A^* = \widetilde{A}$, and we can write the following representation

$$A = S + iT, \tag{58}$$

where $S = \frac{1}{2}(A + \widetilde{A}) = \frac{1}{2}(A + A^*)$, $T = \frac{1}{2i}(A - \widetilde{A}) = \frac{1}{2i}(A - A^*)$. Here operators S and T are J-real and self-adjoint. Since the operator A is unitary, then

$$E = A^*A = (S - iT)(S + iT) = S^2 + T^2 + i(ST - TS).$$

From this relation it follows that operators T and S commute and

$$S^2 + T^2 = E. \tag{59}$$

Since operators S and T are commuting bounded self-adjoint operators, then they admit the following spectral resolutions

$$S = \int_L \lambda dE_\lambda, \qquad T = \int_L z dF_z, \tag{60}$$

where E_λ, F_z are commuting resolutions of unity of operators, and $L = (l_1, l_2]$, $l_1, l_2 \in \mathbb{R}$, is a finite interval of the real line, which contains the spectra of operators. Moreover, since operators S and T are J-real, then their resolutions of unity commute with J. By equality (59) and using spectral resolutions we get

$$\int_L \int_L (\lambda^2 + z^2 - 1) dE_\lambda dF_z = 0, \tag{61}$$

where the integral means a limit in the norm of H of the corresponding Riemann-Stieltjes type sums. Thus, in all points of increase of the measure $dE_\lambda dF_z$ the following relation is true

$$\lambda^2 + z^2 - 1 = 0. \tag{62}$$

A circle (62) in the plane $\mathbb{R}^2$ we denote by Γ. For all points of the circle Γ it is true $|z| = \sqrt{1 - \lambda^2}$ (where we mean the arithmetic value of the root). Therefore for all points of Γ

$$z = \operatorname{sgn}(z)\sqrt{1 - \lambda^2}, \tag{63}$$

where $\operatorname{sgn}(\cdot)$ is from (48). By the identity $\lambda = \cos \arccos \lambda$, $\lambda \in [-1, 1]$, the equality (63) can be rewritten in the following form

$$z = \operatorname{sgn}(z)\sqrt{\sin^2(\arccos \lambda)} = \operatorname{sgn}(z)\sin(\arccos \lambda), \tag{64}$$

where we have used the positivity of sine function on $[0,\pi]$. By this representation we can write

$$A = S + iT = \int_L \int_L (\lambda + iz) dE_\lambda dF_z = \int_L \int_L (\cos\arccos\lambda + i\mathrm{sgn}(z) *$$

$$*\sin(\arccos\lambda)) dE_\lambda dF_z = \int_{L_+} \int_L e^{i\arccos\lambda} dE_\lambda dF_z + \int_{L_-} \int_L e^{-i\arccos\lambda} dE_\lambda dF_z, \tag{65}$$

where $L_+ = (0,\infty) \cap L,\ L_- = (-\infty, 0] \cap L$. Define the following operator

$$S := \int_L \int_L \mathrm{sgn}(z) \arccos\lambda dE_\lambda dF_z = \int_L \mathrm{sgn}(z) dF_z \int_L \arccos\lambda dE_\lambda. \tag{66}$$

It is obvious that S is a J-real self-adjoint operator. From relation (65) it is seen that (57) is true. $\Box$

Using the proven lemma we shall establish the following theorem.

Theorem 2.2 *Let A be a unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A = Re^{iS}, \tag{67}$$

where R is J-real unitary operator in H, and S is a bounded J-real self-adjoint operator in H.

Proof. Consider an operator A such as in the statement of the Theorem. Suppose that representation (67) is true. Then $A^* = e^{-iS}R^*$

$$\widetilde{A^*} = \widetilde{e^{-iS}}\widetilde{R^*} = J(\cos S - i\sin S)J\widetilde{R^*} = (\cos S + i\sin S)\widetilde{R^*} = e^{iS}R^*,$$

since S and R are J-real. Since R is unitary, we can write

$$\widetilde{A^*}A = e^{iS}R^*Re^{iS} = e^{2iS}. \tag{68}$$

Now we shall not suppose that representation (67) holds true. Since the operator A is unitary, then operators $A^{-1} = A^*$, JA^*J and $G := \widetilde{A^*}A$ are unitary, as well. The operator G is J-self-adjoint since $G^* = A^*\widetilde{A} = J\widetilde{A^*}AJ = \widetilde{G}$. Applying to this operator Lemma 2.2 we find J-real self-adjoint operator S such that

$$G = e^{2iS}. \tag{69}$$

Now we set by definition

$$R = Ae^{-iS}. \tag{70}$$

The operator R is unitary as a product of two unitary operators. Then we can write $\widetilde{R^*} = Je^{iS}A^*J = e^{-iS}\widetilde{A^*}$, and therefore

$$\widetilde{R^*}R = e^{-iS}\widetilde{A^*}Ae^{-iS} = e^{-iS}Ge^{-iS} = E.$$

Since the range of a unitary operator R is the whole H, then by the latter equality we get $\widetilde{R^*} = R^{-1}$. Thus, the operator R is J-unitary. Since the operator R is unitary and J-unitary, it is J-real. From (70) it follows the representation (67). $\Box$

Let A be a linear bounded operator in a Hilbert space H and J be a conjugation in H. It is easy to verify that operators $A^TA = JA^*JA$, $AA^T = AJA^*J$ are bounded J-self-adjoint operators. If $A^TA = AA^T$, then the operator A we shall call **J-normal**. It is clear that bounded J-self-adjoint, J-skew-self-adjoint and J-unitary operators are J-normal. The following theorem is true:

Theorem 2.3 *Let A be a linear bounded operator in a Hilbert space H and $0 \notin \sigma(A)$. Let J be a conjugation in H. Suppose that the spectrum of the operator AA^T has an empty intersection with a radial ray $L_\varphi = \{z \in \mathbb{C} : z = xe^{i\varphi},\ x \geq 0\}$ ($\varphi \in [0, 2\pi)$) in the complex plane. Then the operator A admits a representation*

$$A = SU, \tag{71}$$

where S is a bounded J-self-adjoint operator in H, and U is a bounded J-unitary operator in H. Here

$$S = \sqrt{AA^T}, \tag{72}$$

where the square root is understood according to the Riss calculus. Operators U and S commute if and only if the operator A is J-normal. Moreover, the operator A admits a representation

$$A = U_1S_1, \tag{73}$$

where U_1 is a bounded J-unitary operator in H, and $S_1 = \sqrt{A^TA}$ is a bounded J-self-adjoint operator in H. Operators U_1 and S_1 commute if and only if A is J-normal.

In particular, representations (71) and (73) are true for operators

$$A = E + K, \tag{74}$$

where K is a compact operator in H, $\|K\| < 1$.

Proof. Consider an operator A such as in the statement of the Theorem. We set by definition

$$S = \sqrt{AA^T} = \int_\Gamma \sqrt{\lambda} R_\lambda(AA^T) d\lambda. \tag{75}$$

A contour Γ is constructed in the following way. Let $T_R = \{z \in \mathbb{C} : \ |z| = R\}$ be a circle, which contains $\sigma(AA^T)$ inside, $R > 0$. Let $d > 0$ be a distance between a closed set $\sigma(AA^T)$ and a segment $[0, Re^{i\varphi}]$, where φ is from the statement of the Theorem. Consider parallel segments on the distance $\frac{d}{2}$ of the above segment, join them by a half of a circle in a neighborhood of zero and completing the contour with a part of big crcle T_R, it is not hard to construct a contour Γ, which contains the spectrum of the operator AA^T inside, but do not contain the ray L_φ inside. We choose and fix an arbitrary analytic branch of the root in $\mathbb{C} \backslash L_\varphi$.

A bounded operator $B := AA^T$ is J-self-adjoint, as it was noticed above. Consequently, its resolvent is also a J-self-adjoint operator. In fact, we can write

$$R_\lambda^*(B) = ((B - \lambda E)^{-1})^* = (B^* - \overline{\lambda} E)^{-1} = (\widetilde{B} - \overline{\lambda} E)^{-1} =$$

$$= (J(B - \lambda E)J)^{-1} = J(B - \lambda E)^{-1} J = J R_\lambda(B) J, \quad \lambda \in \rho(B).$$

The operator S is J-self-adjoint, as a limit of J-self-adjoint integral sums. Moreover, there exists an inverse operator S^{-1}, which is also J-self-adjoint. Set

$$U = S^{-1} A, \tag{76}$$

and notice that $U^{-1} = A^{-1} S$ (recall that $0 \notin \sigma(A)$). Then

$$U\widetilde{U^*} = S^{-1} A\widetilde{A^*}\widetilde{(S^{-1})^*} = S^{-1} S^2 S^{-1} = E.$$

Multiplying the latter equality from the left side by U^{-1} we get

$$\widetilde{U^*} = U^{-1}.$$

Thus, the operator U is J-unitary.

Suppose now that in representation (71) operators U and S commute. Then

$$AA^T = SU\widetilde{(U^*)}\widetilde{(S^*)} = S^2,$$

$$A^T A = \widetilde{(U^*)} SSU = S^2.$$

Conversely, if operators A and A^T commute, then using last relations (without the latter equality) we write:

$$S^2 = \widetilde{(U^*)}S^2U = U^{-1}S^2U,$$
$$US^2 = S^2U. \tag{77}$$

Since U commutes with S^2, then it commutes with an arbitrary function of this operator. In particular, U commutes with S.

We shall now establish a possibility of resolution (73) for the operator A. First of all we notice that for an arbitrary linear bounded operator D in H we can write

$$JR_\lambda^*(D)J = J(D^* - \overline{\lambda}E)^{-1}J = (JD^*J - \lambda E)^{-1} = R_\lambda(D^T), \quad \lambda \in \rho(D).$$

Therefore

$$\rho(D) = \rho(D^T), \tag{78}$$

for an arbitrary linear bounded operator D in H. Using this equality for operators A and AA^T we conclude that $0 \notin \sigma(A^T)$ and the ray L_φ does not intersect with the spectrum of the operator A^TA. Applying the proven part of the Theorem with the operator A^T, we shall get a resolution $A^T = SU$, where $S = \sqrt{A^TA}$ is a bounded J-self-adjoint operator, U is a bounded J-unitary operator. Therefore

$$A = \widetilde{U^*}\widetilde{S^*} = U^{-1}S,$$

and it remains to notice that U^{-1} is a bounded J-unitary operator.

If the operator A has the form (74), then $0 \notin \sigma(A)$ and

$$AA^T = (E+K)J(E+K^*)J = E + C, \tag{79}$$

where $C := K + JK^*J + KJK^*J$. Notice that the operator C is compact as a sum of compact operators. The operator $J(E+K^*)^{-1}J(E+K)^{-1}$, as it is easy to see, is the inverse operator for the operator AA^T. Therefore $0 \notin \sigma(AA^T)$. Since the spectrum of a compact operator C is discrete, having a unique point of concentration 0, one can find a ray which is required in the statement of the Theorem. $\square$

3 Matrix representations of J-symmetric and J-skew-symmetric operators.

We shall now turn to a study of matrix representations of J-symmetric and J-skew-symmetric operators. Properties which are analogous to the properties

of symmetric operators are valid here. Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be an orthonormal basis in H, which corresponds to J. Let A be a linear operator in H, which is J-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$.

Define a matrix of the operator A in the basis $\mathcal{F}$: $A_M := (a_{i,j})_{i,j\in\mathbb{Z}_+}$, $a_{i,j} = (Af_j, f_i)$. It is not hard to verify that this matrix is complex symmetric (skew-symmetric) in the case of J-symmetric (respectively J-skew-symmetric) operator A. Notice that the columns of this matrix are square summable, i.e. belong to l^2.

It is known that for an arbitrary linear operator A in a Hilbert space H, in the case when the set $D(A) \cap D(A^*)$ is densee in H, the action of the operator A is given by a matrix multirlication [13]. In particular, it is true for symmetric operators. As far as we know, for other classes of operators a possibility to describe the action of the operator as a matrix multiplication was not established earlier. This property possess J-symmetric and J-skew-symmetric operators, as it shows the following theorem.

Theorem 3.1 *Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be an orthonormal basis in H, which corresponds to J. Let A be a linear operator in H, which is J-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$. Let $A_M = (a_{i,j})_{i,j\in\mathbb{Z}_+}$ be a matrix of the operator A in the basis $\mathcal{F}$. Then*

$$Ag = \sum_{i=0}^{\infty} y_i f_i, \quad y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \; g = \sum_{k=0}^{\infty} g_k f_k \in D(A). \tag{80}$$

Proof. Let us verify the validity of the statement of the Theorem for J-skew-symmetric operator. For the case of J-symmetric operator the proof is analogous. Choose an arbitrary element $g = \sum_{k=0}^{\infty} g_k f_k \in D(A)$. Using that the matrix A_M is skew-symmetric and using relation (5) we write

$$y_i = (Ag, Jf_i) = -(Af_i, Jg) = -(\sum_{k=0}^{\infty} (Af_i, f_k) f_k, \sum_{l=0}^{\infty} \overline{g_l} f_l) =$$

$$= -\sum_{k=0}^{\infty} (Af_i, f_k) g_k = -\sum_{k=0}^{\infty} a_{k,i} g_k = \sum_{k=0}^{\infty} a_{i,k} g_k.$$

□

Let us find out, how strong the matrix A_M of the operator A (considered above) determines the operator A. Since J-symmetric and J-skew-symmetric

operators admit closures, which are also J-symmetric (respectively J-skew-symmetric) operators, we shall already suppose that the operator A is closed. By the matrix A_M one can define, as a matrix multiplication, an operator T on $L := \mathrm{Lin}\,\mathcal{F}$. It is easy to check that this operator is J-symmetric (J-skew-symmetric) in the case of J-symmetric (respectively J-skew-symmetric) operator A. This operator admits a closure $\overline{T}$, which is also a J-symmetric (J-skew-symmetric) operator. If $A = \overline{T}$, then the basis $\mathcal{F}$ we shall call **a basis of the matrix representation** of the operator A.

A question appears: If for every complex symmetric (skew-symmetric) semi-infinite matrix B with square summable columns there exists a J-symmetric (respectively J-skew-symmetric) operator A such that the matrix B will be a matrix of the operator in a corresponding to J basis $\mathcal{F}$, and also $\mathcal{F}$ will be a basis of the matrix representation for the operator A? The answer on this question is affirmative.

Theorem 3.2 *Let an arbitrary complex semi-infinite symmetric (skew-symmetric) matrix $M = (m_{i,j})_{i,j\in\mathbb{Z}_+}$ with columns in l^2 is given. Then there exist a Hilbert space H, a conjugation J in H, a J-symmetric (respectively J-skew-symmetric) operator in H, a corresponding to J orthonormal basis $\mathcal{F}$ in H, $\mathcal{F} \subset D(A)$, such that the matrix M is a matrix of the operator A in the basis $\mathcal{F}$ and $\mathcal{F}$ is a basis of the matrix representation for A.*

Proof. For an arbitrary complex semi-infinite symmetric (skew-symmetric) matrix M with columns in l^2 it is enough to choose an arbitrary Hilbert space H, an arbitrary orthonormal basis $\mathcal{F}$ in it and to define a conjugation in H by formula (2). Then, by using the described above procedure, one constructs an operator $\overline{T}$, which is the required operator. □

Notice that, if $\mathcal{F}$ is a basis of the matrix representation for a closed J-symmetric (J-skew-symmetric) operator A, then *$\mathcal{F}$ will be a basis of the matrix representation for the J-adjoint operator $\widetilde{A} = JAJ$, as well.* In fact, the operator $\widetilde{A}$ is J-symmetric (respectively J-skew-symmetric) by Proposition 1.9. From the continuity of the operator J it follows that $\widetilde{A}$ is closed. Then if we choose an arbitrary element $x \in D(\widetilde{A})$, then $Jx \in D(A)$ and there exists a sequence $\widehat{x}_n \in L := \mathrm{Lin}\{f_k\}_{k\in\mathbb{Z}_+}$, $n \in \mathbb{Z}_+$: $\widehat{x}_n \to Jx$, $A\widehat{x}_n \to AJx$, $n \to \infty$. But then we have $J\widehat{x}_n \in L$, $J\widehat{x}_n \to x$, $JA\widehat{x}_n = \widetilde{A}J\widehat{x}_n \to JAJx = \widetilde{A}x$, $n \to \infty$.

The following theorem is true:

Theorem 3.3 *Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ is a corresponding to J orthonormal basis in H. Suppose that A is a closed J-symmetric (J-skew-symmetric) operator in H, $\mathcal{F} \subset D(A)$, and $\mathcal{F}$ is a basis of*

the matrix representation for the operator A. Let $a_{i,j} = (Af_j, f_i),\ i,j \in \mathbb{Z}_+$. Define an operator B in the following way:

$$Bg = \sum_{i=0}^{\infty} y_i f_i, \qquad y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \quad g = \sum_{k=0}^{\infty} g_k f_k \in D_B, \tag{81}$$

on a set $D_B = \{g = \sum_{k=0}^{\infty} g_k f_k \in H:\ \sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} a_{i,k} g_k|^2 < \infty\}$.
Then $A \subseteq A^T = B$ (respectively $A \subseteq -A^T = B$).

Without conditions that A is closed and $\mathcal{F}$ is a basis of the matrix representation for A, one can only state that $A \subseteq A^T \subseteq B$ (respectively $A \subseteq -A^T \subseteq B$).

Proof. The proof will be given in the case of J-skew-self-adjoint operator A. The case of J-symmetric operator is considered analogously. We first show that $-A^T = -(\widetilde{A})^* \subseteq B$. Choose an arbitrary $g \in D(-(\widetilde{A})^*)$ and set $-(\widetilde{A})^* g = g^*$. Let $g = \sum_{k=0}^{\infty} g_k f_k$, $g^* = \sum_{i=0}^{\infty} \widehat{y_i} f_i$. We can write

$$\widehat{y_i} = (g^*, f_i) = (-(\widetilde{A})^* g, f_i) = -(g, \widetilde{A} f_i) = -(\sum_{k=0}^{\infty} g_k f_k, \sum_{j=0}^{\infty} (\widetilde{A} f_i, f_j) f_j) =$$

$$= -\sum_{k=0}^{\infty} g_k \overline{(\widetilde{A} f_i, f_k)} = -\sum_{k=0}^{\infty} a_{k,i} g_k = \sum_{k=0}^{\infty} a_{i,k} g_k, \quad i \in \mathbb{Z}_+.$$

Therefore $\sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} a_{i,k} g_k|^2 < \infty$ and, hence, we get $g \in D_B$. Also we have $-(\widetilde{A})^* g = g^* = Bg$. Thus, we obtain an inclusion $-(\widetilde{A})^* \subseteq B$. Here we did not use that A is closed and that $\mathcal{F}$ is a basis of the matrix representation for A. The inclusion $A \subseteq -(\widetilde{A})^*$ is obvious.

Let us prove the inclusion $B \subseteq -A^T$. As it was shown above, the operator $\widetilde{A}$ is closed and $\mathcal{F}$ is a basis of the matrix representation for $\widetilde{A}$, as well. Choose an arbitrary $g \in D_B,\ g = \sum_{k=0}^{\infty} g_k f_k$. Using the fact that the matrix of the operator A is skew-symmetric, we write

$$(\widetilde{A} f_i, g) = (\sum_{j=0}^{\infty} (\widetilde{A} f_i, f_j) f_j, \sum_{k=0}^{\infty} g_k f_k) = \sum_{k=0}^{\infty} (\widetilde{A} f_i, f_k) \overline{g_k} =$$

$$= \sum_{k=0}^{\infty} \overline{a_{k,i} g_k} = -\sum_{k=0}^{\infty} \overline{a_{i,k} g_k} = -\overline{\sum_{k=0}^{\infty} a_{i,k} g_k} = -\overline{y_i}, \quad i \in \mathbb{Z}_+;$$

$$(Bg, f_i) = y_i, \quad i \in \mathbb{Z}_+.$$

Therefore

$$-(\widetilde{A}f_i, g) = \overline{(Bg, f_i)} = (f_i, Bg),$$

and

$$-(\widetilde{A}f, g) = (f, Bg), \qquad f \in \mathrm{Lin}\{f_k\}_{k\in\mathbb{Z}_+} =: L.$$

For an arbitrary $f \in D(\widetilde{A})$ there exists aa sequence $\{f^k\}_{k\in\mathbb{Z}_+}, f^k \in L$: $f^k \to f$, $\widetilde{A}f^k \to \widetilde{A}f$, as $k \to \infty$. Passing to the limit as $k \to \infty$ in the equality

$$-(\widetilde{A}f^k, g) = (f^k, Bg)$$

and using the continuity of the scalar product, we obtain

$$-(\widetilde{A}f, g) = (f, Bg), \qquad f \in D(\widetilde{A}).$$

Thus, we have $g \in D((\widetilde{A})^*)$ $(\widetilde{A})^*g = -Bg$. Therefore we get an inclusion $B \subseteq -(\widetilde{A})^*$. $\square$

Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be a corresponding to J orthonormal basis in H. Let A be a closed J-symmetric (J-skew-symmetric) operator in H and $\mathcal{F} \subset D(A)$. Set $a_{i,j} = (Af_j, f_i)$, $i,j \in \mathbb{Z}_+$, and define an operator B by formula (81). Is the operator B J-symmetric (J-skew-symmetric)? We first notice that the domain of an operator $\widetilde{B} = JBJ$ is a set

$$D(\widetilde{B}) = \{h = \sum_{k=0}^{\infty} h_k f_k \in H : \sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} \overline{a_{i,k}} h_k|^2 < \infty\}.$$

If $h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B})$, then

$$\widetilde{B}h = \sum_{i=0}^{\infty} (\sum_{k=0}^{\infty} \overline{a_{i,k}} h_k) f_i.$$

Choose an arbitrary elements $g = \sum_{k=0}^{\infty} g_k f_k \in D_B$ $h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B})$. Using relations (21),(22) it is easy to check that the operator B is J-symmetric (J-skew-symmetric), if the following equalities are true (for all $g \in D_B, h \in D(\widetilde{B})$)

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{i,k} g_k \overline{h_i} = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} a_{i,k} g_k \overline{h_i}.$$

In the latter case, the last theorem can be applied with the operator B to obtain that the operator B is J-self-adjoint (J-skew-self-adjoint).

A question appears about existence of a basis of the matrix representation for a closed J-symmetric (J-skew-symmetric) operator. For an arbitrary closed operator there exists an orthonormal basis in which the operator is a closure of its values on the linear span of the basis (see the proof for symmetric operators in [12], which is valid in the general case, as well). A difficulty in the case of J-symmetric (J-skew-symmetric) operators is that this new basis can be a basis which does not correspond to the conjugation J. So, this question remains open.

4 A structure of the null set.

Consider an arbitrary Hilbert space H. Let J be a conjugation in H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be a corresponding to J orthonormal basis in H. Let us study the set $H_{J;0}$, which we defined above ($H_{J;0} = \{x \in H :\ [x,x]_J = 0\}$). Set

$$H_R := \{x \in H :\ (x, f_k) \in \mathbb{R},\ k \in \mathbb{Z}_+\}. \tag{82}$$

Notice that for an arbitrary element $x \in H$ we can write a resolution:

$$x = x_R + i x_I, \qquad x_R, x_I \in H_R. \tag{83}$$

Namely, if $x = \sum_{k=0}^{\infty} x_k f_k$, we set $x_R := \sum_{k=0}^{\infty} \operatorname{Re} x_k f_k$, $x_I := \sum_{k=0}^{\infty} \operatorname{Im} x_k f_k$. It is easy to see that representation (83) is unique.

Define the following vectors:

$$f_{k,l}^{+} := \frac{1}{\sqrt{2}}(f_k + i f_l), \quad f_{k,l}^{-} := \frac{1}{\sqrt{2}}(f_k - i f_l), \qquad k,l \in \mathbb{Z}_+. \tag{84}$$

The following theorem holds true.

Theorem 4.1 *Let H be a Hilbert space and J be a conjugation in H. Let $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ be a correesponding to J orthonormal basis in H. The set $H_{J;0}$ has the following properties:*
1. *The set $H_{J;0}$ is closed;*
2. $x \in H_{J;0} \Rightarrow Jx \in H_{J;0},\ \alpha x \in H_{J;0},\ \alpha \in \mathbb{C}$*;*
3. $x, y \in H_{J;0} :\ x \perp_J y \Rightarrow \alpha x + \beta y \in H_{J;0},\ \alpha, \beta \in \mathbb{C}$*;*
4. $H_{J;0} = \{x \in H :\ x = x_R + i x_I,\ x_R, x_I \in H_R,\ \|x_R\| = \|x_I\|,\ (x_R, x_I) = 0\}$*;*
5. *The set $H_{J;0}$ has no inner points;*
6. $\operatorname{span} H_{J;0} = H$*;*
7. *A set $\{f_{2k,2k+1}^{+}, f_{2k,2k+1}^{-}\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis in H which elements belong to $H_{J;0}$.*

Proof. The 1-st statement of the Theorem follows from the continuity of the operator J and from the continuity of the scalar product in H.
The second and third statements follows from the linearity of the J-form and from the properties of the conjugation J.
The 4-th statement is directly verified.
Suppose that the set $H_{J;0}$ has an inner point x_0 such that

$$x \in H,\ \|x - x_0\| < \varepsilon \ \Rightarrow\ x \in H_{J;0}, \tag{85}$$

for a number $\varepsilon > 0$. Let us write for x_0 the resolution (83):

$$x_0 = x_{0,R} + ix_{0,I}, \qquad x_{0,R}, x_{0,I} \in H_R. \tag{86}$$

Suppose first that $x_{0,I} \neq 0$. Set

$$x_\varepsilon := x_0 + i\frac{\varepsilon}{2\|x_{0,I}\|}x_{0,I} = x_{0,R} + ix_{0,I}\left(1 + \frac{\varepsilon}{2\|x_{0,I}\|}\right). \tag{87}$$

Notice that $\|x_\varepsilon - x_0\| = \frac{\varepsilon}{2} < \varepsilon$, and, thus, by (85), we obtain that $x_\varepsilon \in H_{J;0}$. Using the proven fourth statement of the Theorem for points x_0 and x_ε, we get

$$\|x_{0,R}\| = \|x_{0,I}\|, \tag{88}$$

and

$$\|x_{0,R}\| = \|x_{0,I}\left(1 + \frac{\varepsilon}{2\|x_{0,I}\|}\right)\| = \|x_{0,I}\| + \frac{\varepsilon}{2} > \|x_{0,I}\|,$$

respectively. The obtained contradiction proves statement 5 for the case $x_{0,I} \neq 0$.

If $x_{0,I} = 0$, then by the fourth statement of the Theorem the relation (88) is true and therefore $x_0 = 0$. But if zero is an inner point of the set $H_{J;0}$, then by the proven second statement of the Theorem we get $H_{J;0} = H$. But it is a nonsense, since, for example, elements of the basis $\mathcal{F}$ do not belong to the set $H_{J;0}$.

Let us prove the seventh statement of the Theorem. Using orthonormality of elements f_k, $k \in \mathbb{Z}_+$, it is directly verified that elements of the set $\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+}$, are orthonormal. Notice that

$$f_{2k} = \frac{1}{\sqrt{2}}(f^+_{2k,2k+1} + f^-_{2k,2k+1}), \quad f_{2k+1} = \frac{1}{\sqrt{2}i}(f^+_{2k,2k+1} - f^-_{2k,2k+1}), \qquad k \in \mathbb{Z}_+. \tag{89}$$

Therefore $\mathrm{span}\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+} = H$ and a set $\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis in H. It remains to notice that

$$[f^\pm_{2k,2k+1}, f^\pm_{2k,2k+1}]_J = \frac{1}{2}[f_{2k} \pm if_{2k+1}, f_{2k} \pm if_{2k+1}]_J = 0,$$

and therefore $f^{\pm}_{2k,2k+1} \in H_{J;0},\ k \in \mathbb{Z}_+$.

The sixth statement of the Theorem follows from the proven seventh statement. $\square$

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On a J-polar decomposition of a bounded operator and matrix representations of J-symmetric, J-skew-symmetric operators.

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In this work a possibility of a decomposition of a bounded operator which acts in a Hilbert space H as a product of a J-unitary and a J-self-adjoint operators is studied, J is a conjugation (an antilinear involution). Decompositions of J-unitary and unitary operators which are analogous to decompositions in the finite-dimensional case are obtained. A possibility of a matrix representation for J-symmetric, J-skew-symmetric operators is studied. Also, some simple properties of J-symmetric, J-antisymmetric, J-isometric operators are obtained, a structure of a null set for a J-form is studied.
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